Lévy processes on quantum groups and examples

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Topics

1. Lévy processes
2. Locally compact quantum groups
3. Rieffel deformations

Lévy processes

G – locally compact group

Definition

A G-valued Lévy process is a family $X = (X_t)_{t\geq 0}$ of random variables from a probability space $(\Omega, \mathbb{P})$ to G such that:

1. $X_0 \equiv e$;
2. X has independent and stationary increments ($X_{s,t} := X_s^{-1} X_t$ for $0 \leq s \leq t$);
3. X is continuous.

Lévy process have been well-studied in probability ($G = \mathbb{R}, \mathbb{R}^n$, Lie groups, $\ldots$) and in algebraic/compact QG theory.

X is called symmetric if $X_t \stackrel{\text{distr}}{=} X_t^{-1}$ for all $t \geq 0$.

Lévy processes

X – Lévy process
$(\mu_t)_{t\geq 0}$:= X's family of distributions:
μ_t is the probability measure on G given by $\mu_t(E) := \mathbb{P}\left(X_t^{-1}(E)\right)$.

Definition

A convolution semigroup of probability measures on G is a family $(\mu_t)_{t\geq 0}$ of probability measures on G satisfying

$$\mu_0 = \delta_e \quad \text{and} \quad \mu_s \star \mu_t = \mu_{s+t} \qquad (\forall s,t \geq 0)$$

and w^*-continuity: $\mu_t(f) \xrightarrow[t\to 0^+]{} \mu_0(f)$ for all $f \in C_0(G)$ $\qquad (\mu(f) := \int_G f \, \mathrm{d}\mu)$.

If X is symmetric, so is $(\mu_t)_{t\geq 0}$ (invariant under inversion)

Lévy processes

Examples on $\mathbb{R}^n$

1. X – Wiener process (= Brownian motion)
   $\mathrm{d}\mu_t(x) = (4\pi t)^{-n/2} \exp\left(-\frac{\|x\|^2}{4t}\right) \mathrm{d}x \ (t > 0)$
2. X – Cauchy process
   $\mathrm{d}\mu_t(x) = \Gamma(\frac{n+1}{2}) t \left[\pi(\|x\|^2 + t^2)\right]^{-\frac{n+1}{2}} \mathrm{d}x \ (t > 0)$

Recall:
Lévy process $\rightsquigarrow$ convolution semigroups $(\mu_t)_{t\geq 0}$ of prob. meas. on G.

$\rightsquigarrow$ (left) convolution operators $(\mu_t \star \cdot)_{t\geq 0}$ on $C_0(G)$ / $L^\infty(G)$.

These operators form UCP semigroup,
and commute with all right convolution operators $\cdot \star \mu$, $\mu \in C_0(G)^*$.

G-valued Lévy processes	$\overset{\text{distr}}{\longleftrightarrow}$	convolution semigroups of prob. measures on G	$\longleftrightarrow$	UCP semigroups on $C_0(G)$ / $L^\infty(G)$ commuting with right convolution operators

Rieffel deformations

Crossed products

C^*-algebra A, locally compact abelian group $G \overset{\rho}{\curvearrowright} A$.

$\rightsquigarrow$ C^*-algebra $A \rtimes_\rho G$ and dual action $\hat{G} \overset{\hat{\rho}}{\curvearrowright} A \rtimes_\rho G$.
$\mathrm{M}(A \rtimes_\rho G)$ contains “copies” of A and G, and for all $a \in A$:

1. $\hat{\rho}$ fixes a;
2. $g \cdot a \cdot g^{-1} = \rho_g(a)$, thus $g \mapsto g \cdot a \cdot g^{-1}$ is norm continuous;
3. certain products of a belong to $A \rtimes_\rho G$ rather than $\mathrm{M}(\cdots)$.

These are the Landstad conditions.

Theorem (Landstad)

The Landstad conditions determine A and $G \overset{\rho}{\curvearrowright} A$.

Rieffel deformations

Landstad deformation of $A \rtimes_\rho G$

$G \overset{\rho}{\curvearrowright} A$ as above, 2-cocycle Φ on $\hat{G}$:

1. $\Phi : \hat{G} \times \hat{G} \to \mathbb{T}$ continuous,
2. $\Phi(e, \cdot) \equiv 1 \equiv \Phi(\cdot, e)$,
3. $\Phi(\hat{g}_1, \hat{g}_2 + \hat{g}_3)\Phi(\hat{g}_2, \hat{g}_3) = \Phi(\hat{g}_1 + \hat{g}_2, \hat{g}_3)\Phi(\hat{g}_1, \hat{g}_2)$.

$\rightsquigarrow$ deformed action $\hat{G} \overset{\hat{\rho}^\Phi}{\curvearrowright} A \rtimes_\rho G$: for $\hat{g} \in \hat{G}$,

- denote $\Phi(\cdot, \hat{g}) \in C_b(\hat{G}) \hookrightarrow \mathrm{M}(A \rtimes_\rho G)$ by $U_{\hat{g}}$;
- set $\hat{\rho}^\Phi_{\hat{g}} := (\mathrm{Ad}\, U^*_{\hat{g}}) \circ \hat{\rho}_{\hat{g}}$.

$\rightsquigarrow$ The elements in $\mathrm{M}(A \rtimes_\rho G)$ satisfying the Landstad conditions for $\hat{\rho}^\Phi$ in lieu of $\hat{\rho}$ form a C^*-algebra A^Φ. We also get an action $G \overset{\rho^\Phi}{\curvearrowright} A^\Phi$.

By Landstad's theorem we have $A \rtimes_\rho G = A^\Phi \rtimes_{\rho^\Phi} G$. The algebra A^Φ is called the Rieffel deformation of A.

Locally compact quantum groups

Motivation

"Take out" commutativity from algebras like $C_0(G), L^\infty(G)$.

Definition (Kustermans–Vaes, '00)

A locally compact quantum group is a pair $\mathbb{G} = (L^\infty(\mathbb{G}), \Delta)$ such that:

1. $L^\infty(\mathbb{G})$ is a von Neumann algebra.
2. $\Delta : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \,\overline{\otimes}\, L^\infty(\mathbb{G})$ is a co-multiplication: a normal, faithful, unital $*$-homomorphism which is co-associative, i.e.,
$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta.$$
3. Haar weights: there exist two n.s.f. weights on $L^\infty(\mathbb{G})$ that are left and right invariant, resp.

Various algebras: $L^\infty(\mathbb{G})$, $C_0(\mathbb{G})$, $C_0^{\mathrm{u}}(\mathbb{G})$.

Locally compact quantum groups

Example 1: $\mathbb{G} = G$

$L^\infty(\mathbb{G}) = L^\infty(G)$, $C_0(\mathbb{G}) = C_0^{\mathrm{u}}(\mathbb{G}) = C_0(G)$
$(\Delta f)(t, s) := f(ts)$.

Example 2: $\mathbb{G} = \hat{G}$

$L^\infty(\mathbb{G}) = \mathrm{VN}(G)$, $C_0(\mathbb{G}) = C^*_{\mathrm{r}}(G)$, $C_0^{\mathrm{u}}(\mathbb{G}) = C^*_{\mathrm{f}}(G)$,
$\Delta\lambda_g := \lambda_g \otimes \lambda_g$.

Locally compact quantum groups

Examples

- Atomic examples
  - usually obtained as a deformation of a group
  - e.g.: $SU_q(n)$, $E_q(2)$, $ax+b$, $az+b$, ...
- Discrete/compact quantum groups:
  - “free”: free unitary/orthogonal quantum groups, quantum symmetric groups
  - quantum automorphism groups
  - “combinatorial”
  - ...
- Constructions
  - double crossed products
  - bicrossed products

Locally compact quantum groups

Convolution semigroups of states on $C_0^{\mathrm{u}}(\mathbb{G})$:

Definition

A convolution semigroup of states on $\mathbb{G}$ is a family $(\mu_t)_{t\geq 0}$ of states on $C_0^{\mathrm{u}}(\mathbb{G})$ satisfying

$$\mu_0 = \epsilon \quad \text{and} \quad \mu_s \star \mu_t = \mu_{s+t} \qquad (\forall s, t \geq 0)$$

and w^*-continuity at 0^+.

Theorem (Daws)

These correspond bijectively to UCP semigroups on $L^\infty(\mathbb{G})/C_0^{\mathrm{u}}(\mathbb{G})/C_0(\mathbb{G})$ commuting with all right convolution operators.

Rieffel deformations of groups

G – locally compact group
abelian $\Gamma \leq G$
2-cocycle Ψ on $\hat{\Gamma}$

- Consider $\Gamma^2 \overset{\text{left-right}}{\curvearrowright} C_0(G)$.
- Rieffel deformation $\rightsquigarrow$ C^*-algebra $C_0(G)^\Psi$ inside $\mathrm{M}(C_0(G) \rtimes \Gamma^2)$.
- Deform the co-mult. on $C_0(G)$ to get a co-mult. on $C_0(G)^\Psi$.

Theorem (Kasprzak, Fima–Vainerman)

The above procedure yields a LCQG G^Ψ
(in the sense of Woronowicz, and under additional assumptions, in the sense of K–V).

Remark: by $C_0(G)^\Psi$ we mean $C_0(G)^\Phi$ with $\Phi : \hat{\Gamma}^4 \to \mathbb{T}$ given by

$$\Phi(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{\gamma}_4) := \overline{\Psi(-\hat{\gamma}_1, -\hat{\gamma}_3)}\Psi(\hat{\gamma}_2, \hat{\gamma}_4).$$

Rieffel deformations of groups

Example

$G := \mathrm{SL}(2, \mathbb{C}) := \{A \in M_2(\mathbb{C}) : \det A = 1\}$,
$\Gamma := \mathbb{C}_* \hookrightarrow \mathrm{SL}(2, \mathbb{C})$ as $\left\{\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{C}_*\right\}$,
$\Psi_s(z_1, z_2) := \exp\left(\mathrm{i}s\Im(z_1\overline{z_2})\right)$ for fixed $s \in \mathbb{R}$.

Rieffel deformation $\rightsquigarrow$

Woronowicz–Zakrzewski quantum deformation(s) of $\mathrm{SL}(2, \mathbb{C})$.

Convolution semigroups on Rieffel deformations

- As before: G – l.c. group, abelian $\Gamma \leq G$, 2-cocycle Ψ on $\hat{\Gamma}$.
- $(\mu_t)_{t\geq 0}$ – convolution semigroup of prob. meas. on G, invariant under the adjoint action of Γ.

Theorem (Skalski–V)

1. *The UCP semigroup induced by $(\mu_t)_{t\geq 0}$ on $C_0(G)$ extends to $C_0(G) \rtimes_{\text{left-right}} \Gamma^2$ and restricts to a UCP semigroup on $C_0(G)^\Psi$.*
2. *This semigroup on $C_0(G)^\Psi$ is induced by a convolution semigroup of states $\left(\mu_t^\Psi\right)_{t\geq 0}$ on G^Ψ.*
3. *If all μ_t's are symmetric (= invariant under inversion), then all μ_t^Ψ's are symmetric (invariant under the [unitary] antipode of G^Ψ).*

Example

Consider the above quantum deformation of $\mathrm{SL}(2,\mathbb{C})$ w.r.t. $\mathbb{C}_* \hookrightarrow \mathrm{SL}(2,\mathbb{C})$.

Measure on $\mathbb{C}_* \rightsquigarrow$ measure on $\mathrm{SL}(2,\mathbb{C})$ invariant under the adjoint action of $\mathbb{C}_*$.

Thus, every (symmetric) Lévy process on $\mathbb{C}_*$ induces a (symmetric) convolution semigroup of states on the deformed $\mathrm{SL}(2,\mathbb{C})$.

Convolution semigroups on LCQGs

Theorem (Skalski–V)

There exist $1-1$ correspondences between:

1. *symmetric convolution semigroups of states on $\mathbb{G}$;*
2. *completely Markov semigroups on $L^\infty(\mathbb{G})$ that are right-translation invariant and KMS-symmetric;*
3. *completely Markov semigroups on $L^2(\mathbb{G})$ that are right-translation invariant and symmetric;*
4. *completely Dirichlet forms that are right-translation invariant.*

Notions (Beurling–Deny, Albeverio–Høegh-Krohn, Sauvageot, Davies–Lindsay, Guido–Isola–Scarlatti, Cipriani–Sauvageot, Cipriani, and finally Goldstein–Lindsay)

- Markov operator = "mapping $[0, \mathbb{1}]$ to itself" (roughly!).
- Markov semigroup = continuous semigroup of such
- KMS-symmetry

More examples: cocycle twistings

Data

$\mathbb{G}$ – LCQG
$\Omega \in \mathcal{U}(L^\infty(\mathbb{G}) \bar{\otimes} L^\infty(\mathbb{G}))$ – 2-cocycle:

$$(\mathbb{1} \otimes \Omega) \cdot (\mathrm{id} \otimes \Delta)(\Omega) = (\Delta \otimes \mathrm{id})(\Omega) \cdot (\Omega \otimes \mathbb{1}).$$

Define $\Delta_\Omega : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \bar{\otimes} L^\infty(\mathbb{G})$ by

$$\Delta_\Omega := \Omega \Delta(\cdot) \Omega^*.$$

Theorem (De Commer)

$(L^\infty(\mathbb{G}), \Delta_\Omega)$ is a LCQG.

This process is called cocycle twisting.
It preserves neither the Haar weights nor compactness!

Remark

Cocycle twisting and the Rieffel deformation are dual to one another!

More examples: cocycle twistings

- As before: $\mathbb{G}$ – LCQG, Ω – 2-cocycle on $\mathbb{G}$.
- $(\mu_t)_{t\geq 0}$ – convolution semigroup of states on $\mathbb{G}$.
- $(T_t)_{t\geq 0}$ – the associated UCP semigroup on $L^\infty(\mathbb{G})$.

Proposition (Skalski–V)

$(T_t \otimes \mathrm{id})(\Omega) = \Omega$ for all $t \implies$
$(T_t)_{t\geq 0}$ arises from a convolution semigroup of states on $\mathbb{G}_\Omega$.

General example

$\mathbb{G} := \hat{G}$ and abelian $\Gamma \leq G$
2-cocycle Ψ on the l.c.a.g. $\hat{\Gamma} \rightsquigarrow$ 2-cocycle Ω on $\mathbb{G}$
A semigroup of normalized positive-definite functions on G which is $\equiv 1$ on Γ satisfies the condition of the proposition $\rightsquigarrow$ convolution semigroup of states on $\hat{G}_\Omega$.

More examples: cocycle twistings

Example (quantum Heisenberg group)

$G = \mathbb{H}_n(\mathbb{R}) = \mathbb{R}^{n+1} \rtimes \mathbb{R}^n =$ the Heisenberg group and $\Gamma = \mathbb{R}^{n+1}$ with a particular cocycle Ω on $\hat{\Gamma}$
$\rightsquigarrow \hat{G}_\Omega$ is the quantized Heisenberg group $\mathbb{H}_n^q(\mathbb{R})$ of Enock–Vainerman.

Thus:
convolution semigroup on $\mathbb{R}^{n+1} \rightsquigarrow$ convolution semigroup on $\mathbb{H}_n^q(\mathbb{R})$,

and if the former is symmetric, so is the latter.

A. Skalski and A. Viselter, work in progress.

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Thank you for your attention!

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Thank You, Paul!