# EXERCISES ON WINDOWS

(1) (a) Convince yourself that $\mathrm{Coh}_{\mathbb{C}^*}(\mathbb{C}^{n+1} \setminus 0)$ is equivalent to $\mathrm{Coh}(\mathbb{P}^n)$.

(b) Using the usual covering by two affine charts, find (all graded pieces of) the Čech cohomology of the structure sheaf on $\mathbb{C}^2 \setminus 0$. Repeat for $\mathbb{C}^n \setminus 0$.
*If you've never computed the cohomology of line bundles on projective space then this exercise is highly recommended.*

(2) Let $\mathcal{A} \subset \mathrm{Coh}_{\mathbb{C}^*}(\mathbb{C}^{n+1})$ be the abelian subcategory generated by $\mathcal{O}, \mathcal{O}(1), ..., \mathcal{O}(n)$. Show that the restriction functor $\rho^* : \mathcal{A} \to \mathrm{Coh}(\mathbb{P}^n)$ is *not* an equivalence (and hence $D^b(\mathcal{A})$ is not the same as $\mathcal{W}$).

(3) (a) Let $\mathbb{C}^*$ act on $\mathbb{C}^3_{x,y,z}$ with weights $1, 1, d$, for some $d \in \mathbb{Z}$. Let $X$ be the quotient of the open set $\{(x, y) \neq (0, 0)\}$ by this action. Convince yourself that $X$ is the total space of the line-bundle $\mathcal{O}(d)$.

(b) Let $\mathcal{W} \subset D^b_{\mathbb{C}*}(\mathbb{C}^3)$ be the subcategory generated by $\mathcal{O}$ and $\mathcal{O}(1)$. If $d > 0$ then the restriction functor $\mathcal{W} \to D^b(X)$ is *not* an equivalence. Why not?

(4) Let $X_+$ and $X_-$ be the two sides of the Atiyah flop, and let $\Phi : D^b(X_+) \to D^b(X_-)$ be the derived equivalence constructed from the window:

$$\mathcal{W} = \big\langle\, \mathcal{O},\, \mathcal{O}(1) \,\big\rangle \quad \subset D^b_{\mathbb{C}*}(\mathbb{C}^4)$$

(a) Find an object in $\mathcal{W}$ whose restriction to $X_+$ is $\mathcal{O}(-1)$. Hence compute $\Phi(\mathcal{O}(-1))$. What are its homology sheaves?

(b) Now let $\Psi : D^b(X_+) \to D^b(X_-)$ be the derived equivalence constructed from the window $\langle \mathcal{O}(-1), \mathcal{O} \rangle$. Show that the autoequivalence $T = \Phi \circ \Psi^{-1}$ is not the identity functor. Can you say anything more about it?

(5) Let $\mathbb{C}^*$ act on $\mathbb{C}^3$ with weights $(1, 1, -1)$. Let $X_\pm$ be the two GIT quotients.

(a) What are $X_+$ and $X_-$?

(b) Let $\mathcal{W} = \langle \mathcal{O}, \mathcal{O}(1) \rangle \subset D^b_{\mathbb{C}*}(\mathbb{C}^3)$. Find an object in $\mathcal{W}$ which restricts to zero on $X_-$.

(c) Show that $D^b(X_+)$ has a semi-orthogonal decomposition into $D^b(X_-)$ and one other object. *This is Orlov's blow-up formula.*

(6) Let $X$ be the total space of the rank two vector bundle $\mathcal{O}(-1, -1)^{\oplus 2}$ over $\mathbb{P}^1 \times \mathbb{P}^1$. It can be constructed as a GIT quotient of $\mathbb{C}^6$ by $(\mathbb{C}^*)^2$ acting with weights:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{pmatrix}$$

(a) Find the unstable strata and the grade-restriction rules. Find four equivariant line bundles on $\mathbb{C}^6$ that satisfy all the grade-restriction rules. Do you think these bundles generate $D^b(X)$?

(b) This GIT problem has three quotients. What are the unstable strata and the grade-restriction rules for the other two quotients? Is there a 'magic' window that works for all three?

(c) What happens if we instead use the follow weight matrix?

$$\begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}$$

*Hint: there are three quotients, and for every quotient there are three unstable strata.*

# References for Windows and GIT

Thomas - Notes on GIT and symplectic reduction for bundles and varieties
https://arxiv.org/abs/math/0512411

Halpern-Leistner – The derived category of a GIT quotient
https://arxiv.org/abs/1203.0276v1
*NB – this link is to v1 of the paper which is more readable than later versions.*

Segal – Equivalences between GIT quotients of Landau-Ginzburg B-models
https://arxiv.org/abs/0910.5534

Ballard-Favero-Katzarkov - Variation of geometric invariant theory quotients and derived categories
https://arxiv.org/abs/1203.6643

Donovan-Segal - Window shifts, flop equivalences and Grassmannian twists
https://arxiv.org/abs/1206.0219

Halpern-Leistner-Sam - Combinatorial constructions of derived equivalences
https://arxiv.org/abs/1601.02030

Halpern-Leistner – Derived Θ-stratifications and the D-equivalence conjecture
https://arxiv.org/abs/2010.01127
*Advanced!*