1. Problems

Bundles on curves.

Problem 1.1. (1) Show that every vector bundle V on $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles $V \simeq \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ for a uniquely determined sequence of integers $a_1 \leq a_2 \leq \cdots \leq a_r$. (Hint: Use induction on the rank. Show that there is a maximal a such that there is a nonzero map $\mathcal{O}_{\mathbb{P}^1}(a) \to V$. Show that the cokernel of this map is a vector bundle. Show that for the given choice of a, there are no nontrivial extensions.)

(2) Conclude that there are no stable bundles on $\mathbb{P}^1$ of rank at least 2 and there exists semistable bundles on $\mathbb{P}^1$ only when the rank divides the degree.

(3) Determine the cohomology of a vector bundle on $\mathbb{P}^1$.

(4) Show that the tangent bundle $T\mathbb{P}^n$ is not a direct sum of line bundles for $n > 1$. If you want a challenge, show that $T\mathbb{P}^n$ is stable.

Problem 1.2. (1) Show that every vector bundle V on a genus 1 curve E is isomorphic to a direct sum of semistable vector bundles (Hint: Show that the Harder-Narasimhan filtration splits.)

(2) Show that there exists a semistable vector bundle on E of every rank and degree. Show that there exists a stable vector bundle on E if and only if the rank and the degree are coprime.

(3) Determine the cohomology of a semistable vector bundle V on E if $\mathcal{O}_E$ does not occur as a Jordan-Hölder factor of V. Give examples of semistable vector bundles on E with the same associated graded but different cohomology groups.

Problem 1.3. Let C be a smooth, projective curve of genus $g \geq 2$.

(1) Construct semistable bundles of every rank $r \geq 1$ and degree d on C. If you want a challenge, show that the general semistable bundle is stable.

(2) Show that the general stable bundle V has no higher cohomology if $\chi(V) \geq 0$ and has no global sections if $\chi(V) \leq 0$.

(3) Show that the general stable bundle V is globally generated if $\chi(V) \geq r + 1$.

(4) Show that if V is a semistable vector bundle with $\mu(V) > 2g - 2$, then V has no higher cohomology. Construct examples of semistable bundles with $\mu(V) \leq 2g - 2$ that have higher cohomology.

(5) Show that if V is a semistable bundle with $\mu(V) > 2g-1$, then V is globally generated. Find examples of semistable bundles with $\mu(V) \leq 2g - 1$ which are not globally generated.

Bundles on $\mathbb{P}^2$.

Problem 1.4. (1) Show that an exceptional sheaf on $\mathbb{P}^n$ is a vector bundle. Give an example of an exceptional sheaf on a Hirzebruch surface $\mathbb{F}_e$ with $e \geq 1$ which is not a vector bundle.

(2) Using the Riemann-Roch Theorem, show that the rank and c_1 of an exceptional bundle on $\mathbb{P}^2$ are relatively prime.

(3) Using the fact that $\chi(E, E) = 1$ for an exceptional bundle on $\mathbb{P}^2$, compute the discriminant in terms of the slope.

(4) Show that on $\mathbb{P}^2$ there is a unique exceptional bundle with a given exceptional slope.

(5) Compute several mutations of the standard exceptional collection $\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2)$.

Problem 1.5. (1) Show that If F is a pure sheaf of rank $r > 0$, then

$$\mu\text{-stability} \implies \text{Gieseker stability} \implies \text{Gieseker semistability} \implies \mu\text{-semistability}.$$

Prove that these notions coincide when $c_1(V) \cdot H^{n-1}$ and $r(V)H^n$ are relatively prime. Show that the reverse implications are false in general (see the next two steps for examples).

(2) Let p be a point in $\mathbb{P}^2$. Show that $\mathcal{O}_{\mathbb{P}^2} \oplus I_p$ is slope semistable but not Gieseker semistable. Prove that there are no Gieseker semistable sheaves of rank 2, $c_1 = 0$ and $\Delta = \frac{1}{2}$. Deduce that an elementary modification of a Gieseker semistable sheaf (hint: consider $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$) may fail to be Gieseker semistable.

(3) Let p, q, r be three distinct points on $\mathbb{P}^2$. Show that $\text{ext}^1(I_p, I_{q,r}) = 2$. Prove that a nonsplit extension

$$0 \to I_{q,r} \to E \to I_p \to 0$$

is Gieseker stable but not slope stable.

Problem 1.6. (1) Show that a general stable bundle V on $\mathbb{P}^2$ admits a resolution (called the Gaeta resolution) of the form

$$0 \to \mathcal{O}_{\mathbb{P}^2}(a-2)^\gamma \to \mathcal{O}_{\mathbb{P}^2}(a-1)^\beta \oplus \mathcal{O}_{\mathbb{P}^2}(a)^\alpha \to V \to 0, \text{ or}$$

$$0 \to \mathcal{O}_{\mathbb{P}^2}(a-2)^\gamma \oplus \mathcal{O}_{\mathbb{P}^2}(a-1)^\beta \to \mathcal{O}_{\mathbb{P}^2}(a)^\alpha \to V \to 0.$$

(2) Compute the Gaeta resolution for the ideal sheaf of 2, 3, 4 or 5 general points on $\mathbb{P}^2$.

(3) Deduce that the moduli space of stable sheaves on $\mathbb{P}^2$ is unirational and that the general member is locally free.

(4) Prove that if V is a general stable bundle with $\mu(V) \geq 0$ and $\chi(V) \geq r + 2$, then V is globally generated.

Bundles on K3 surfaces.

Problem 1.7. Let X be a K3 surface with $\text{Pic}(X) \cong \mathbb{Z}H$ and $H^2 = 2n$.

(1) Show that there exists a spherical bundle V with resolution

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(H)^{n+2} \to V \to 0$$

and compute the cohomology of V.

(2) Let C be a smooth member of $|H|$. Consider extensions of the form

$$0 \to \mathcal{O}_X(H)^{\oplus r} \to V \to \mathcal{O}_C(L) \to 0,$$

where L is a line bundle on C with Euler characteristic $2n - r$. Compute the Mukai vector of V. Determine when the general member of the moduli space has this form. Compute the cohomology of V.

(3) Let $n = 1$ and f_k denote the kth Fibonacci number starting with $f_1 = f_2 = 1$. Show that there is a stable spherical bundle with resolution

$$0 \to \mathcal{O}_X^{\oplus f_{2k-2}} \to \mathcal{O}_X(H)^{\oplus f_{2k}} \to V \to 0.$$

Compute the cohomology of V.

(4) Consider the Mukai vector $(r, rpH, rp^2n - 1)$ where r and p are integers. Show that the general member of the corresponding moduli space is not locally free.

Bundles on more general surfaces.

Problem 1.8. Let X be a surface with a curve C such that $C^2 < 0$. Show that for each rank there exists infinitely many moduli spaces of sheaves of rank r where weak Brill-Noether fails.

Problem 1.9 (Mestrano's example)**.** Let X be a very general sextic surface in $\mathbb{P}^3$. Show that the moduli space of sheaves on X may be reducible by considering extensions of the following form

$$0 \to \mathcal{O}_X \to V \to I_Z(H) \to 0,$$

where Z is a zero-dimensional scheme of length 11 and Z is either contained in a twisted cubic curve or in a hyperplane section of X. Find other examples of reducible and/or nonreduced moduli spaces of sheaves on surfaces.

The stable base locus decomposition and Bridgeland walls on $\mathbb{P}^2$.

Problem 1.10. Assume that $r(\zeta), r(\xi) > 0$ and $\mu(\zeta) \neq \mu(\xi)$. Show that the potential Bridgeland wall $W(\zeta, \xi)$ in the st-plane a semi-circle with center (c,0), where

$$c = \frac{1}{2}(\mu(\zeta) + \mu(\xi)) - \frac{\Delta(\zeta) - \Delta(\xi)}{\mu(\zeta) - \mu(\xi)},$$

and radius ρ with

$$\rho^2 = (c - \mu(\zeta))^2 - 2\Delta(\zeta).$$

Problem 1.11. If an ideal sheaf of n points on $\mathbb{P}^2$ is destabilized along a wall W given by a subobject of rank at least 2, then the radius ρ_W of W satisfies

$$\rho_W^2 \leq \frac{n}{4}.$$

Problem 1.12. Compute the stable base locus decomposition of $\mathbb{P}^{2[n]}$ for $n = 2, 3, 4$. Classify the Bridgeland walls in the st-plane for the ideal sheaves of $2, 3$ or 4 points.

Problem 1.13. Show that the monomial scheme defined by $I_Z = (x^4, xy, y^4)$ has the Betti diagram as a general scheme of length 7. However, show that I_Z is destabilized in a wall bigger than the collapsing wall. Conclude that the same Betti diagram is not sufficient to determine the Bridgeland wall where an ideal sheaf is destabilized.

Problem 1.14. Compute the effective cone of the moduli spaces of sheaves on $\mathbb{P}^2$ with invariants $r = 3$, $\mu = \frac{2}{3}$ and $\Delta = \frac{17}{9}$.

The ample cone on more general surfaces.

Problem 1.15. Let X be a very general surface of degree $d > 3$ in $\mathbb{P}^3$ and let H denote the hyperplane class on $\mathbb{P}^3$. Compute the ample cone of the moduli space of rank 2 vector bundles with $c_1 = H$ and $\Delta \gg 0$.

Problem 1.16. Let X be a double cover of $\mathbb{P}^2$ branched along a very general curve of degree $2d \geq 6$ and let H denote the pullback of the hyperplane class from $\mathbb{P}^2$. Compute the ample cone of the moduli space of rank 2 vector bundles with $c_1 = H$ and $\Delta \gg 0$.