Exercises for the Technion summer school

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I strongly encourage you to work in groups. You probably only have time to do one or two of these exercises, so skim them all and start with the one that appeals to you the most. If you're not sure about definitions or results from the lectures, please ask!

1. Let X be a K3 surface. Then

$$H^*(X, \mathbb{Z}) \;=\; \mathbb{Z} \quad 0 \quad \mathbb{Z}^{22} \quad 0 \quad \mathbb{Z},$$

so

$$K^0_{\text{top}}(X) = \mathbb{Z}^{24} \qquad\qquad K^1_{\text{top}}(X) = 0.$$

The point of this exercise is to prove that the Mukai vector

$$v\colon K^0_{\text{top}}(X) \to H^*(X, \mathbb{Q})$$

is an isomorphism onto $H^*(X, \mathbb{Z})$.

(a) Compute $\operatorname{td}(T_X)$ and $\sqrt{\operatorname{td}(T_X)}$. Use the fact that $c_1(T_X) = 0$ and $c_2(T_X) = 24\xi$, where $\xi \in H^4(X, \mathbb{Z})$ is Poincaré dual to a point. You can find the formula for the Todd class on Wikipedia, or in Appendix A of Hartshorne.

(b) For any class $\kappa \in K^0_{\text{top}}(X)$, show that both

$$\operatorname{ch}(\kappa) \qquad \text{and} \qquad v(\kappa) := \operatorname{ch}(\kappa)\sqrt{\operatorname{td}(T_X)}$$

are in $H^*(X, \mathbb{Z})$, not just $H^*(X, \mathbb{Q})$. Use the fact that the intersection pairing is even: if $\alpha \in H^2(X, \mathbb{Z})$, then $\alpha^2 \in 2\mathbb{Z}\xi$.

(c) For every i and every $\alpha \in H^i(X, \mathbb{Z})$, find a class $\kappa \in K^0_{\text{top}}(X)$ such that $v(\kappa) = \alpha$. Use the fact that for every $\alpha \in H^2(X, \mathbb{Z})$, there is a unique topological line bundle L with $c_1(L) = \alpha$, which is true for any X with the homotopy type of a finite CW-complex.

(d) Convince yourselves that you have proved the initial claim.

(e) Convince yourselves that the same claim does not hold for a cubic surface.

2. (If you really like topology.) Let X be an Abelian n-fold, so topologically it is a torus $(S^1)^{2n}$. The point of this exercise is to prove that the Mukai vector

$$v\colon K^*_{\mathrm{top}}(X) \to H^*(X, \mathbb{Q})$$

is an isomorphism onto $H^*(X, \mathbb{Z})$.

(a) Observe that the tangent bundle is a trivial bundle, so $\mathrm{td}(T_X) = 1$, so the Mukai vector is the same as the Chern Character.

(b) The claim holds for the circle using the corollaries to the Atiyah–Hirzebruch spectral sequence that I gave on Monday: we have

$$H^*(S^1, \mathbb{Z}) \ = \ \mathbb{Z} \quad \mathbb{Z},$$

so

$$K^0_{\mathrm{top}}(S^1) = \mathbb{Z} \qquad\qquad K^1_{\mathrm{top}}(S^1) = \mathbb{Z},$$

and the leading term of the Chern character of any class is integral, but in this case there are only leading terms.

Or you can see it directly: for $K^0(S^1)$, observe that complex vector bundles on S^1 are all trivial; you might deduce this from the fact that $\mathrm{GL}_n(\mathbb{C})$ is connected. For $K^1(S^1) = K^{-1}(S^1)$, identify it with the reduced $\tilde{K}^0(\Sigma S^1) = \tilde{K}^0(S^2) = \tilde{K}^0(\mathbb{CP}^1)$ and mess around with the rank-0 classes $\mathcal{O}_{\mathbb{P}^1}(k) - \mathcal{O}_{\mathbb{P}^1}$.

(c) There is a Künneth formula for the topological K-theory of a product, just like the familiar one for the cohomology of a product. The oldest reference is probably Atiyah, "Vector bundles and the Künneth formula," but you can search online for more recent references and the compatibility with the Chern character. Convince yourselves that this proves the claim on $(S^1)^{2n}$.

(d) Alternatively, you can use the fact that the suspension of a product of spheres is a bouquet of spheres: more generally,

$$\Sigma(X \times Y) = \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y),$$

where $\vee$ is the wedge product and $\wedge$ is the smash product – for a nice picture, see Hatcher, Proposition 4I.1 – and we have $S^m \wedge S^n = S^{m+n}$. So to prove the claim on products of cirlces, we can prove it on reduced K-theory of all spheres. Work out the details.

(e) Now you can easily prove a generalization of Corollary 9.24 in Huybrechts' Fourier–Mukai book.

3. Let $X = \mathbb{P}^2$, and let $E = I_\Delta$ be the ideal sheaf of the diagonal $\Delta \subset X \times X$.

   (a) Show that $\Phi_E(\mathcal{O}_X) = 0$.

   (b) Convince yourselves that the exact sequence

   $$0 \to I_\Delta \to \mathcal{O}_{X\times X} \to \mathcal{O}_\Delta \to 0$$

   induces an exact triangle

   $$\Phi_E(F) \to \mathcal{O}_X \otimes \mathrm{R}\Gamma(F) \to F$$

   for all $F \in D(X)$.

   (c) Show that $\Phi_E(\mathcal{O}_X(-1)) = \mathcal{O}_X(-1)[-1]$, and similarly with $\mathcal{O}_X(-2)$.

   (d) On Monday I asserted that the action of Φ_E^H on $H^*(X, \mathbb{Q}) = \mathbb{Q}[h]/h^3$ is given by

   $$\begin{aligned} 1 &\mapsto -\tfrac{25}{32} + \tfrac{21}{128}h + \tfrac{49}{1024}h^2 \\ h &\mapsto \tfrac{3}{4} - \tfrac{7}{16}h + \tfrac{21}{128}h^2 \\ h^2 &\mapsto 1 + \tfrac{3}{4}h - \tfrac{25}{32}h^2. \end{aligned}$$

   Check that this agrees with the facts above, using

   $$v(\Phi_E(F)) = \Phi_E^H(v(F))$$

   and

   $$\sqrt{\mathrm{td}(T_X)} = 1 + \tfrac{3}{4}h + \tfrac{7}{32}h^2$$

   If you think you've found a mistake in my computation, let me know.

4. Let X be a (smooth complex projective) curve of genus 1, let $\Delta \subset X \times X$ be the diagonal, and let $E = \mathcal{O}_{X\times X}(-\Delta)$. In my first lecture, I argued that the functor

$$\Phi_E \colon D^b(X) \to D^b(X)$$

takes

$$\mathcal{O}_x \mapsto \mathcal{O}_X(-x), \qquad \mathcal{O}_X \mapsto \mathcal{O}_X[-1], \qquad \mathcal{O}_X(x) \mapsto \mathcal{O}_x[-1],$$

where if $x \in X$ is a point then $\mathcal{O}_x$ denotes its skyscraper sheaf and $\mathcal{O}_X(\pm x)$ is a line bundle of degree ± 1. The point of this exercise is to finish the details of the last claim.

(a) To compute $\Phi_E(\mathcal{O}_X(x))$, we want to take the line bundle

$$L := p^*\mathcal{O}_X(x) \otimes E = \mathcal{O}_{X\times X}(x \times X - \Delta).$$

and apply Rq_*. (Draw a picture!)
Over a point $y \in X$, the restriction of L to the fiber $q^{-1}(y)$ is the line bundle $\mathcal{O}_X(x - y)$, which has $h^0 = h^1 = 1$ if $x = y$, and $h^0 = h^1 = 0$ if $x \neq y$, and of course $h^{\geq 2} = 0$. From Hartshorne, Chapter III Theorem 12.11, we conclude that $R^{\geq 2}q_*L = 0$, and that R^1q_*L is supported at x, and its restriction to the point x has rank 1. Similarly we find that R^0q_*L vanishes away from x, but we get no control of the rank of its restriction to x.

(b) Thus $R^1q_*L = \mathcal{O}_{mx}$ for some $m \geq 1$, meaning the structure sheaf of the fat point of length m supported at x. (There is only one such fat point because X is a smooth curve. It is isomorphic to $\operatorname{Spec}(\mathbb{C}[t]/t^{m+1})$.) Similarly, $R^0q_*L = \mathcal{O}_{n_1x} \oplus \cdots \oplus \mathcal{O}_{n_kx}$ for some $n_1, \ldots, n_k$. Both follow from the structure theorem for finitely generated modules over the principal ideal domain, applied to the local ring $\mathcal{O}_{X,x}$.

(c) Now the Leray spectral sequence for q gives

$$h^0(L) = n_1 + \cdots + n_k, \qquad\qquad h^1(L) = m$$

(d) On the other hand, we have seen that

$$R^0p_*E = 0, \qquad\qquad R^1p_*E = \mathcal{O}_X,$$

so the projection formula (as reviewed in Sunday's lecture, or use Hartshorne, Chapter III Exercise 8.3) gives

$$R^0p_*L = 0, \qquad\qquad R^1p_*L = \mathcal{O}_X(x),$$

so the Leray spectral sequence for p gives

$$h^0(L) = 0, \qquad\qquad h^1(L) = 1.$$

Convince yourselves that you've finished proving $\Phi_E(\mathcal{O}_X(x)) = \mathcal{O}_x[-1]$.

(e) If you're feeling very energetic, adapt this argument to show that $E \circ E^* = \mathcal{O}_\Delta[-1] = E^* \circ E$.